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A MODULE CONSTRUCTION FOR A CLASS FUNCTION ON A FINITE GROUP

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Abstract: In this article, we study a class function of a finite group. By computing the inner product, we prove the class function is a character, and a kind of its module construction is obtained.

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1 Introduction

In [4], J.G.Thompson considered some interesting generalized characters on a finite group G. For example, for an element $g \in G$, he defined

$$\chi_p(g) = |\{h \in G : \text{the Sylow } p-\text{subgroups of } <g,h> \text{ are abelian}\}|.$$

In [4], J.G.Thompson proved that χ_p is a generalized character of G.

In this note, we consider a class function f defined on a finite group and prove that it is a character; by the regular representation of the center of the complex group algebra, we give a kind of module construction of the class function.

2 Definitions and Basic Results

Definition 1 Let G be a finite group, and the finite group $G \times G$ be the direct product of G and G. We define a function f on G by

$$f(g) = |\{(u,v) \in G \times G : g = [u,v] = u^{-1}v^{-1}uv\}|,$$

here g is any element of the finite group G and for a set S , $|S|$ denotes the cardinality of S.

Lemma 1 f is a class function on G . An element g which is equal to some $[u,v], u, v \in G$ is called a commutator. If $g \in G$ is not a commutator, then $f(g) = 0$.

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Proof The statements are obvious by the definition of f.

In our main theorem of this note, we prove that even f is a character of G.

Now, we fix some notations for this note. Throughout this note, G always denotes a finite group. By $\mathrm{Irr}(G) = \{\chi_1 = 1_G, \cdots, \chi_k\}$, we denote the set of all nonequivalent complex irreducible characters of G, where 1_G is the unity character for G. It's well-known that the degree $\chi_i(1)$ of any irreducible character is a divisor of $|G|$. For any $i \in \{1, \cdots, k\}$, let M_i denote the corresponding simple $\mathbf{C}G$-module affording the character χ_i and T_i denote the corresponding representation, where $\mathbf{C}$ is the complex number field. Here a character of G is always afforded by a $\mathbf{C}G$-module.

Definition 2 We call $R(G) = \{\sum\limits_{i=1}^{i=k} a_i\chi_i, a_i \in \mathbf{Z}\}$ the character ring of the finite group G, here $\mathbf{Z}$ is the ring of rational integers. $R(G)$ is a commutative ring under the addition and the multiplication of functions.

Lemma 2 For any $\phi, \psi \in R(G)$, we have the usual inner product$(\text{-} , \text{-})_G$ on $R(G)$ defined as follows

$$(\phi, \psi)_G = \frac{1}{|G|}\sum_{g\in G}\phi(g)\psi(g^{-1}).$$

Furthermore, $(\chi_i, \chi_i)_G = 1$, while $(\chi_i, \chi_j)_G = 0$ if i is different from j, i.e., $\mathrm{Irr}(G)$ is a standard orthogonal basis of $R(G)$.

3 Main Results and Proofs

To obtain a module construction for the function f defined in Definition 1, we only need to compute the numbers $m_i = (f, \chi_i)_G, i = 1, \cdots, k$. Indeed, if we know all the m_i(we will see below that any m_i is a positive integer), then the module corresponding to f is isomorphic to $\bigoplus_{i=1}^{i=k} m_iM_i$, and then the $\mathbf{C}G$-module $\bigoplus_{i=1}^{i=k} m_iM_i$ affords the character f.

Before computing the numbers m_i, $i = 1, \cdots, k$, we will present a key lemma which is well-known and appeared in [2] first; see also the proof of Theorem 30′ in Chapter 5 in [1].

Theorem 1 For any $\chi \in \mathrm{Irr}(G)$, by T we denote the corresponding representation affording the character χ. For a fixed element $u \in G$, the following equation holds:

$$\sum_{v\in G} T(v^{-1}uv) = \tau(u)I,$$

here $\tau(u) = \big(\frac{|G|}{\chi(1)}\big)\chi(u)$ is a complex number depending on u and I is the $\chi(1)\times\chi(1)$ identity matrix.

Proof For any $g \in G$, it's easy to see that

$$T(g)\Big(\sum_{v\in G} T(v^{-1}uv)\Big)T(g^{-1}) = \sum_{v\in G} T(v^{-1}uv),$$

which means that $\sum\limits_{v\in G} T(v^{-1}uv)$ commutes with any $T(g)$, then by Schur's lemma, we get that $\sum\limits_{v\in G} T(v^{-1}uv) = \tau(u)I$, where $\tau(u)$ is a complex number and I is the $\chi(1)\times\chi(1)$ identity matrix. Taking the traces of both sides of the above equality, noting that χ is a class function on G, we obtain $\tau(u) = \left(\frac{|G|}{\chi(1)}\right)\chi(u)$.

We are done for the lemma.

Theorem 2 If f is the class function on G defined in Definition 1, then for any $i \in \{1, \cdots, k\}$, $m_i = (f, \chi_i)_G = \frac{|G|}{\chi_i(1)}$ is a positive integer. f is a character of G and the module affording f is isomorphic to $\bigoplus_{i=1}^{i=k} \frac{|G|}{\chi_i(1)} M_i$.

Proof Replace T by T_i in Theorem 1. Multiplying $T_i(u^{-1})$ on both sides of the equality in the proof of Theorem 1 , taking sum over $u \in G$ and noting that T_i is a representation for G, we get

$$\sum_{u\in G}\sum_{v\in G} T_i(u^{-1}v^{-1}uv) = \frac{|G|}{\chi_i(1)} \sum_{u\in G} \chi_i(u) T_i(u^{-1}),$$

and then we take traces of both sides. Putting $g = u^{-1}v^{-1}uv = [u, v]$, we obtain

$$\sum_{g\in G} f(g)\chi_i(g) = \frac{|G|}{\chi_i(1)} \sum_{u\in G} \chi_i(u)\chi_i(u^{-1}) = \frac{|G|^2}{\chi_i(1)}(\chi_i, \chi_i)_G = \frac{|G|^2}{\chi_i(1)}.$$

What's more, by the definition of f, it's evidently that $f(g) = f(g^{-1})$; indeed if $[u, v] = g$, then $[v, u] = g^{-1}$. Thus through rewriting the left side of the foregoing equality, we get

$$(f, \chi_i)_G = (\chi_i, f)_G = \frac{1}{|G|} \sum_{g\in G} f(g)\chi_i(g) = \frac{|G|}{\chi_i(1)}.$$

And the other statements in the theorem follow easily.

From now on, we have proven all of our statements and obtained a module construction for character f defined in Definition 1. In the following part of this note, we will give a module construction of f by the regular representation of the center of $\mathbf{C}G$ and some consequences of the main result.

Corollary 1 For any $g \in G$, from what we obtained we see that the number of all the different $(u, v) \in G \times G$, such that $[u, v] = g$ equals $\sum\limits_{i=1}^{i=k} \frac{|G|}{\chi_i(1)}\chi_i(g)$. Consequently, Burnside's theorem on commutators follows: An element $g \in G$ is a commutator if and only if $\sum\limits_{i=1}^{i=k} \frac{\chi_i(g)}{\chi_i(1)}$ is not equal to 0; see [1, Chapter 3, Theorem 26].

Next, we handle $f(g)$ in another direction. Let $Z(\mathbf{C}G)$ denote the center of the group algebra $\mathbf{C}G$. Let $cl(G) = \{C_1, \cdots, C_k\}$ be the set of conjugacy classes of the finite group G and $g_j \in C_j$ be a representative of the conjugacy class C_j. Then, the following statements are well-known.

Lemma 3 The class sums $\hat{C}_i = \sum\limits_{x\in C_i} x, i = 1, \cdots, k$, form a basis of $Z(\mathbf{C}G)$ and the functions $\omega_{\chi_i} : Z(\mathbf{C}G) \to \mathbf{C}^*$, $i = 1, \cdots, k$, cover all the irreducible representations of $Z(\mathbf{C}G)$, where $\mathbf{C}^*$ is the multiplicative group of all non-zero complex numbers, and for any $C_j \in cl(G)$, $\omega_{\chi_i}(\hat{C}_j) = \frac{|C_j|\chi_i(g_j)}{\chi_i(1)}$.

Now, we can make the function f in Definition 1 more clear. By using the regular representation of the center of the group algebra, we obtain the following corollary.

Corollary 2 For any $g \in G$, if we define $\beta_g = \sum\limits_{x \in G} x^{-1}gx$, then $f(g)$ defined in Definition 1 is just the value of the regular character R of $Z(\mathbf{C}G)$ at β_g.

Proof By the results above, we see that

$$f(g_j) = \sum_{i=1}^{i=k} \frac{|G|}{\chi_i(1)} \chi_i(g_j) = |C_G(g_j)| \sum_{i=1}^{i=k} \omega_{\chi_i}(\hat{C}_j),$$

where $C_G(g_j)$ represents the centralizer of g_j in G. Let R denote the regular character for $Z(\mathbf{C}G)$, then we find that, for any $j \in \{1, \cdots, k\}$, $f(g_j) = |C_G(g_j)| R(\hat{C}_j) = R(\beta_g)$.

Furthermore, if we write $\hat{C}_j \hat{C}_i = \sum\limits_{l=1}^{l=k} t_{jil} \hat{C}_l$, where t_{jil} are nonnegative integers for all $i, j, l \in \{1, \cdots, k\}$; see [3, Subsection 3.2..5]. Thus, we get that $f(g_j) = |C_G(g_j)| \sum\limits_{i=1}^{i=k} t_{jii}$.

Corollary 3 If g_j is not a commutator, then all $t_{jii} = 0$.

Proof Because all the $t_{ijl}, i, j, l \in \{1, \cdots, k\}$ are nonnegative integers and $f(g_j) = 0$, by the equality above, we know that all $t_{jii} = 0$.

If we note that $t_{jil} = |\{(a, b) \in C_i \times C_j : ab = g_l\}|$, then we can show Corollary 3 easily by elementary group theory.

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有限群的一个类函数的模构造

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摘 要: 本文研究了有限群上的一个类函数. 通过计算它和不可约特征标的内积, 证明了它是特征标并且通过复群代数的中心的正则表示给出了它的一个模构造.

关 键 词: 换位子; 特征标; 类函数

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